Soliton motion in a parametrically ac-driven damped Toda lattice

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We demonstrate that a staggered parametric ac driving term can support stable progressive motion of a soliton in a Toda lattice with friction, while an unstaggered driving force cannot. A physical context of the model is that of a chain of anharmonically coupled particles adsorbed on a solid surface of a finite size. The ac driving force is generated by a standing acoustic wave excited on the surface. Simulations demonstrate that the state left behind the moving soliton, with the particles shifted from their equilibrium positions, gradually relaxes back to the equilibrium state that existed before the passage of the soliton. The perturbation theory predicts that the ac-driven soliton exists if the amplitude of the drive exceeds a certain threshold. The analytical prediction for the threshold is in reasonable agreement with that found numerically. Collisions between two counterpropagating solitons is also simulated, demonstrating that the collisions are, effectively, fully elastic. [S1063-651X(98)07111-6]

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The important role of collective nonlinear excitations in the form of solitons in several condensed-matter physics contexts is widely recognized [1]. Particularly, discrete nonlinear models have received an increased amount of attention lately [2]. Actual observation of nontrivial dynamical effects in realistic systems is often impeded by friction. An externally supplied driving force is therefore necessary to sustain a soliton in power balance with friction. In the simplest case, progressive motion of a soliton in the presence of dissipation is supported by a dc driving force [3], which in fact is the only possibility in a homogeneous continuum medium. A less obvious but physically important option is to drive a soliton by an ac force (with no dc component) [4]. It has been done in a continuum system subjected to a periodic spatial modulation [5], and in a discrete system in the form of a chain of nonlinearly interacting particles [6,7].

A simple example of the latter system is the ac-driven damped Toda lattice (TL)

$$\ddot{x}_n+\exp(x_n-x_{n-1})-\exp(x_{n+1}-x_n)$$
$$=-\alpha\dot{x}_n+(-1)^n\epsilon\sin(\omega t), \tag{1}$$

where x_n is the displacement of the nth particle, ϵ and ω are the amplitude and frequency of the driving force, and α is the friction coefficient. This model describes a chain of nonlinearly coupled particles in a dissipative environment driven by an externally applied electric ac field. The alternating sign in front of the driving term, $(-1)^n$ (a discrete pattern of this type is frequently called *staggered* [8]), implies alternating electrical charges of the particles in the chain; without the multiplier $(-1)^n$, the driving term can be trivially eliminated from Eq. (1). In Ref. [6], it was demonstrated analytically and numerically that the model (1) indeed supports progressive motion of a soliton in the presence of finite friction (this was accomplished for this model in the so-called dual form). ac-driven motion was predicted and found at the *resonant* soliton velocities

$$V=\omega/\pi(2\nu+1),\quad \nu=0,\pm1,\pm2,\ldots, \tag{2}$$

i.e., the velocity of the ac-driven motion is directly controlled by the driving frequency.

The objective of the present work is to study moving solitons in a similar but essentially different model, viz., a *parametrically* ac-driven damped TL, with the driving taken both in the staggered form,

$$\ddot{x}_n+\exp(x_n-x_{n-1})-\exp(x_{n+1}-x_n)$$
$$=-\alpha\dot{x}_n+(-1)^n\epsilon\sin(\omega t)x_n, \tag{3}$$

and in the unstaggered one, differing from Eq. (3) by the absence of the multiplier $(-1)^n$. The model (3) describes a physical system in the form, for example, of a chain of particles adsorbed on a solid surface and interacting through an anharmonic repulsive potential. Additionally, the particles interact with a uniform external field aligned perpendicular to the adsorbing surface. (The field may simply be gravitation, or a dc electric field.) In the latter case, all the particles are assumed to have the same charge (and the interaction between them within the chain is generated by the charges through Coulomb repulsion). The effective drive is induced by a standing acoustic (elastic) wave excited on the surface, the end particles in the chain being fixed at the surface edges. If the length of the surface is L, and the number of particles is $N+1$ then the repelling particles are separated by the distance L/N in the equilibrium state. The finite size of the surface selects the wavelengths of the standing elastic waves $\lambda=2L/M$, where M is an arbitrary integer. Commensurability between the adsorbed chain and the standing wave takes place provided $M=pN$, where $p=1,2,3,\ldots$. If the commensurability index p is odd, starting from $p=1$, the distance between neighboring particles in the adsorbed chain is equal to an odd number of the standing half-waves, hence the ac driving forces acting upon the neighboring particles are π

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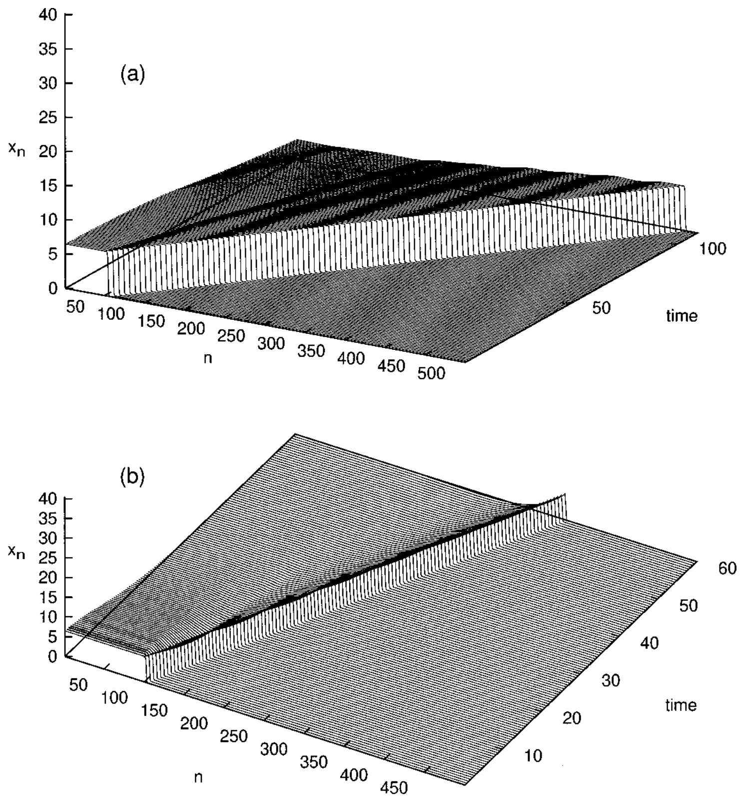

FIG. 1. Typical examples of a moving front with the staggered driving force: (a) weak friction ($\alpha = 0.01$) and (b) relatively strong friction ($\alpha = 0.18$). In both cases, $\epsilon = 0.2$ and $\omega = 12.4$.

out of phase, yielding the staggered form of the driving term in Eq. (3). Oppositely, when p is even, the phase shift between the neighboring particles is a multiple of 2π corresponding to the unstaggered driving force. Finally, it is necessary to specify the boundary conditions for the standing elastic wave. If the edges of the surface are fixed, the particles in the adsorbed chain are located at the nodes (zero-amplitude points) of the standing wave. It is easy to see that in this case the external field oriented perpendicular to the surface gives rise to the direct driving-force term in Eq. (1). On the other hand, if the edges are free, the particles are located at the standing-wave maximum-amplitude points, which gives rise to the *parametric* drive in Eq. (3) (for sufficiently small x_n).

In the following we will first numerically demonstrate that the model (3) is able to sustain propagating solitonlike solutions whose velocity is locked by the frequency of the drive. Further, we demonstrate that only in the presence of the *staggered* driving force are the solitonlike solutions supported. Locking of the velocity to the driving frequency, and the fact that the solitonlike solutions need a staggered drive to exist, is then explained analytically and the threshold of the driving amplitude is found. Finally the relaxation dynamics of the particles in the wake of the soliton is studied analytically.

The system of equations (1) was solved numerically for a chain consisting of 500 particles. The natural initial condition is the exact soliton solution of the unperturbed TL,

$$x_n(t) = -\ln\left[1 + \frac{(\xi^{-2}-1)}{[1+\xi^{-2(n-n_0-Vt)}]}\right], \tag{4}$$

where the real parameter ξ $(-1<\xi<1)$ determines the properties of the soliton (width and velocity), and n_0 is an arbitrary phase constant. The velocity V is

$$V = (\xi - 1/\xi)/\ln(\xi^{-2}). \tag{5}$$

Launching the soliton (4) into the system (3) gives rise to scenarios such as those depicted in Figs. 1(a) and 1(b). In the case of Fig. 1(a), the friction is very weak ($\alpha = 0.01$), and the initial kink transforms into a propagating front. In the wake of the front, the particles demonstrate relatively large oscillations, slowly relaxing to their equilibrium positions. In the case of stronger friction ($\alpha = 0.18$), Fig. 1(b) shows that the front is still created, but the oscillations of the particles in its wake are strongly overdamped, so that they relax rapidly to the initial positions. In the latter case, the net shift in the position of each particle generated by the passing front is zero, making the driven front a more local phenomenon than the soliton of the unperturbed TL.

The situation is quite different when the unstaggered driving force is applied. Figure 2 shows a representative example of the motion of a driven front in this case. The front is seen to form in much the same fashion as in the model with the staggered drive. However, the front is gradually decreasing its amplitude and eventually ceases to exist. Nevertheless, the decay time may be rather long, up to 100 periods of the ac drive.

Since the front in the staggered model behaves similarly to a soliton, it is interesting to launch counterpropagating fronts and observe their collision. An example is shown in Fig. 3, where it is clearly seen that the fronts survive the collision. A closer study of trajectories of the two fronts shows that the velocity is unchanged by the collision and, more surprisingly, *no phase shift* occurs.

We now investigate the observed phenomenon analytically and specify the region of its existence in parameter space. As in [6], the analytical consideration will be based on the perturbation theory, assuming α and ϵ to be sufficiently

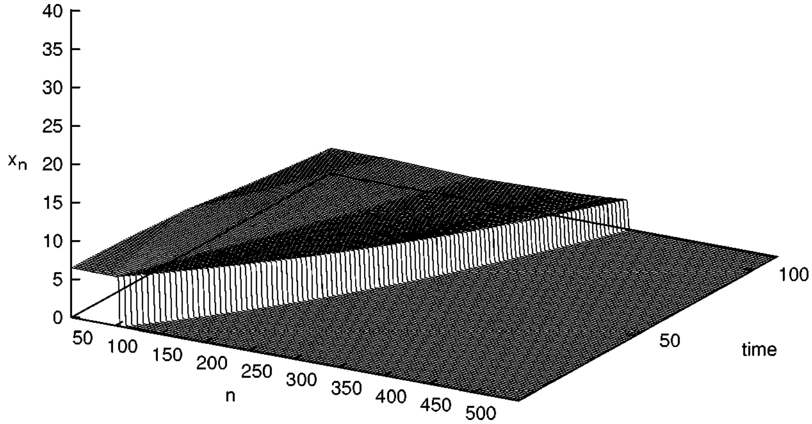

FIG. 2. A representative example of the moving front in the case of the unstaggered driving force. The parameters are $\alpha=0.02$, $\epsilon=0.2$, and $\omega=12.4$.

small. In the zeroth-order approximation, the exact soliton solution to the unperturbed TL equation is given in Eq. (4). Passage of the soliton through a given particle gives rise to its displacement:

$$\Delta x_n=\int_{-\infty}^{+\infty}\dot{x}_n(t)dt=\mathrm{sgn}\,\xi\ln(\xi^{-2}). \tag{6}$$

If the drive is taken in the staggered form, as in Eq. (3), the analysis similar to that of Ref. [6] shows that the ac-driven motion of the soliton is expected at the same resonant velocities (2). The unstaggered drive gives rise to the other spectrum of the resonant velocities,

$$V=\omega/2\pi\nu,\quad \nu=\pm1,\pm2,\ldots. \tag{7}$$

The goal of the analytical approach is to predict the minimum (threshold) value $\epsilon_{\rm thr}$ of the ac-drive amplitude ϵ that is sufficient to compensate the friction and allows progressive motion of the soliton. Each value of the integer ν in Eqs. (2) and (7) should thus give rise to a corresponding function $\epsilon_{\rm thr}(\alpha,\omega)$. Steady motion of the soliton through the damped driven lattice is possible if the momentum lost by each particle under the action of the friction force, $-\alpha\dot{x}_n$, is balanced by the momentum input from the driving force, $(-1)^n\epsilon\sin(\omega t)x_n$. The momentum loss of the particle can be calculated as

$$(\Delta P)_{\rm loss}=-\alpha\int_{-\infty}^{+\infty}\dot{x}_n(t)dt=-\alpha\,\mathrm{sgn}\,\xi\ln(\xi^{-2}), \tag{8}$$

where Eq. (6) was used. The momentum input is

$$(\Delta P)_{\rm input}=(-1)^n\epsilon\int_{-\infty}^{+\infty}\sin(\omega t)X_n(t)dt$$
$$\equiv(-1)^n\epsilon\omega^{-1}\int_{-\infty}^{+\infty}\cos(\omega t)\dot{X}_n(t)dt, \tag{9}$$

where integration by parts was performed and the contribution from $t\to\pm\infty$ was neglected (since it is zero on average). The integral in Eq. (9) can be calculated explicitly, inserting the expression for $\dot{X}_n(t)$ following from the unperturbed soliton solution (4). The result depends on the arbitrary phase constant n_0; the threshold is determined by equating the *largest* possible absolute value of the momentum input (9) to the absolute value of the momentum loss (8). After a simple calculation, this leads to the equality

$$\alpha\ln(\xi^{-2})=\frac{2\pi}{\omega}\epsilon_{\rm thr}\frac{|\sin\{\frac{1}{2}[\omega/(\xi-1/\xi)]\ln(\xi^{-2})\}|}{\sinh[\pi\omega/(\xi-1/\xi)]}. \tag{10}$$

Replacing the combination $(\xi-1/\xi)^{-1}\ln(\xi^{-2})$ by what follows from Eq. (5), and using the resonant relations (2) or (7), one immediately concludes that in the case (7), corresponding to the unstaggered drive, the factor multiplying $\epsilon_{\rm thr}$ in Eq. (10) vanishes. This implies that the progressive motion of the soliton cannot be supported by the unstaggered ac drive in the presence of the friction. This agrees with our numerical observations. However, substitution of Eq. (2) for the staggered driving leads to a finite threshold,

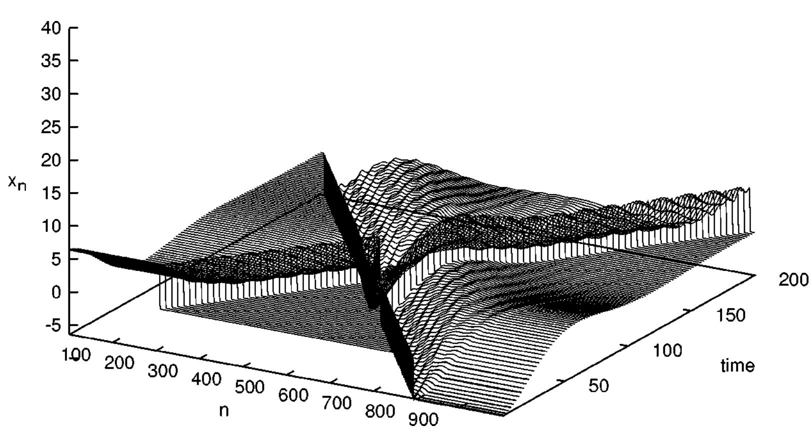

FIG. 3. A typical example of the quasielastic collision between two ac-driven solitons. The values of the parameters are $\epsilon=0.2$, $\alpha=0.1$, and $\omega=12.4$.

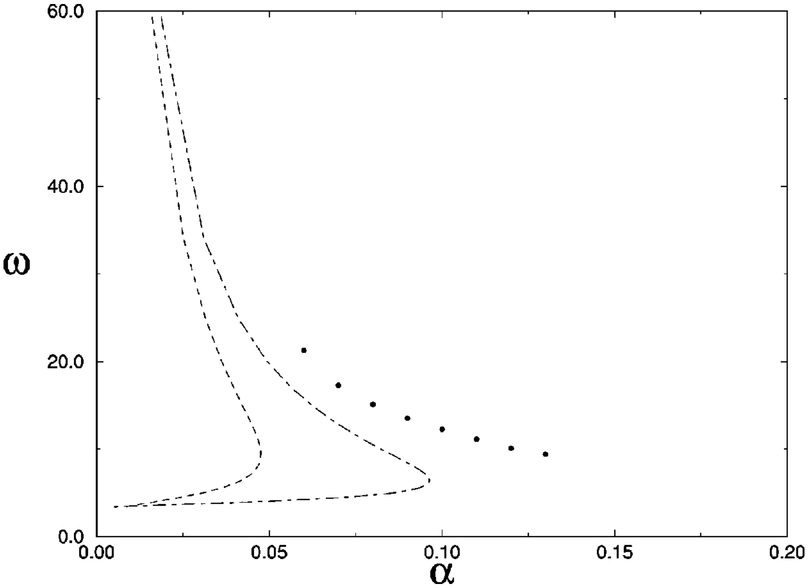

FIG. 4. The dependence between the driving frequency ω and the maximum friction coefficient α at which a stable moving soliton can be supported by the ac drive with the fixed amplitude $\epsilon=0.2$. The dots are results of direct numerical simulations of Eq. (3). The dash-dotted and dashed curves represent, respectively, the analytical results provided by the full perturbative approximation based on Eqs. (11) and (12), and by the simplified approximation (15). The parts of the curves beneath the turning points are irrelevant (see text).

$$\epsilon_{\text{thr}}=\frac{\alpha\omega}{2\pi}\ln(\xi_0^{-2})\sinh\left(\frac{\pi\omega}{\xi_0-1/\xi_0}\right)\equiv\alpha f(\omega), \tag{11}$$

where the parameter ξ_0 is determined, as a function of ν and ω, by a transcendental equation following from Eqs. (2) and (5):

$$\pi(2\nu+1)(\xi_0-1/\xi_0)=\omega\ln(\xi_0^{-2}). \tag{12}$$

Note that the linear relationship between ϵ_{thr} and α in Eq. (11) is an evident feature of the first-order approximation of the perturbation theory [6]. The function ϵ_{thr} is given implicitly by Eqs. (11) and (12). To obtain an explicit dependence, one needs to solve Eq. (12) for ξ_0, which cannot be done analytically, but an approximate solution is given by

$$\xi_0^{-1}\approx\frac{2\omega}{\pi(2\nu+1)}\ln\left[\frac{2\omega}{\pi(2\nu+1)}\right], \tag{13}$$

provided that the driving frequency is sufficiently large, so that

$$\ln\left[\frac{2\omega}{\pi(2\nu+1)}\right]\gg\ln\left[\ln\left(\frac{2\omega}{\pi(2\nu+1)}\right)\right]. \tag{14}$$

In this limit, expression (11) takes an approximate explicit form,

$$\epsilon_{\text{thr}}=\frac{\alpha\omega}{\pi}\ln\left(\frac{2\omega}{\pi(2\nu+1)}\right)\times\sinh\left[\frac{1}{2}\pi^2(2\nu+1)/\ln\left(\frac{2\omega}{\pi(2\nu+1)}\right)\right]. \tag{15}$$

In order to verify the analysis we have compared the simulations with the analytical results, and found that the propagating soliton can be created and sustained at resonant velocities given by Eq. (2) when the soliton parameter ξ is chosen in accordance with Eq. (12). Further, we have found that the soliton velocity locks to the driving frequency even when the initial velocity deviates slightly from the resonance-locked value. That is, the simulations demonstrate that if the initial velocity fulfills $(\omega-\delta\omega)/\pi<V<(\omega+\delta\omega)/\pi$, where $\delta\omega\simeq3$, the soliton will be created and sustained at the resonant velocity.

The threshold of the driven motion was measured numerically in the following way. At a fixed value of the driving amplitude ϵ, the simulations were run at different values of ω, gradually increasing the friction coefficient until reaching a maximum value $\alpha_{\max}$, beyond which the drive could no longer support the stable moving soliton. In terms of Eq. (11),

$$\alpha_{\max}(\epsilon,\omega)=\epsilon/f(\omega), \tag{16}$$

this procedure implies a way to numerically measure the function $f(\omega)$.

In Fig. 4, the numerically obtained dependence $\alpha_{\max}(\omega)$ is plotted, along with the analytically predicted dependence (16), at the fixed value of the amplitude, $\epsilon=0.2$. The analytical dependence is shown in two forms: the simplified one as per Eq. (15) (dashed line), and the more accurate form (dashed-dotted line) obtained by numerically solving the transcendental equation (12) and substituting the result into Eq. (11). The parts of both analytical curves beneath the turning points are irrelevant: a direct inspection shows that they correspond either to unphysical solutions with $\xi^2>1$, or to very broad solitons, for which, in fact, the model becomes overdamped, and the perturbation theory is invalid. A conclusion suggested by Fig. 4 is that, although the accuracy provided by the perturbation theory is not very high, it captures the systematics reasonably. It is also noteworthy that the simplified approximation (15) is much worse than the full perturbative approximation based on Eq. (12). This is not surprising: even for the largest frequency, $\omega\equiv22$, for which the result of the direct simulations is given in Fig. 1, the condition (14), necessary for applicability of the simplified approximation, takes the form $2.63\gg0.97$.

Finally, the slow relaxation of the oscillations behind the passing soliton can easily be analyzed. Assuming the oscillation amplitude to be small enough, one can linearize Eq. (3), yielding

$$\ddot{x}_n-(x_{n-1}-2x_n+x_{n+1})=(-1)^n\epsilon\sin(\omega t)x_n-\alpha x_n. \quad (17)$$

A solution to Eq. (17) can be sought as a sum of a rapidly oscillating staggered component and a slowly relaxing unstaggered one,

$$x_n(t)=(-1)^n x(t)+X(t). \quad (18)$$

Substituting this into Eq. (17) and collecting the staggered and unstaggered terms, we obtain two simple equations:

$$\ddot{x}+\alpha\dot{x}+4x=\epsilon X\sin(\omega t), \quad (19)$$

$$\ddot{X}+\alpha\dot{X}=\epsilon\sin(\omega t)x. \quad (20)$$

Using the fact that the friction is much weaker than the inertia, and following the assumption according to which the unstaggered part of the solution is slowly relaxing [in comparison with the rapid oscillations of $\sin(\omega t)$], one can readily obtain the following approximate solution to Eq. (19):

$$x(t)\approx-\epsilon(\omega^2-4)^{-1}\sin(\omega t)X(t). \quad (21)$$

Next, substituting Eq. (21) into Eq. (20) and replacing the rapidly oscillating term with its averaged value, we obtain an effective equation governing the slow relaxation of the state left behind the passing soliton:

$$\ddot{X}+\alpha\dot{X}+\tfrac{1}{2}\epsilon^2(\omega^2-4)^{-1}X=0. \quad (22)$$

Depending on the relation between the small parameters α and $\epsilon^2(\omega^2-4)^{-1}$, this equation may describe both oscillatory and aperiodic relaxation to the final state $X=0$, which is the equilibrium state that existed before the passage of the soliton. Furthermore, the relaxation frequency specified by Eq. (22) is very close to the relaxation frequency observed in the simulations.

In summary, we have demonstrated, analytically and numerically, that a staggered parametric ac driving term can support stable progressive motion of solitons in a damped Toda lattice, while an unstaggered drive cannot. Also, the threshold condition for the existence of the ac-driven soliton predicted by the perturbation theory is in reasonable agreement with that found numerically. The simulations demonstrate that the state left behind the moving soliton gradually relaxes back to the equilibrium state that existed before the passage of the soliton, which we easily explained analytically. Finally, we demonstrated that collisions between two solitons moving with opposite velocities are nearly elastic.

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